Define $\boldsymbol{v} \equiv \boldsymbol{v}_j$ and expand the quantity $\langle E(v)\boldsymbol{v}\rangle$ in Eq. (11) to get

$$\langle E(v)\boldsymbol{v}\rangle \equiv \langle E(v)\boldsymbol{v}_j\rangle = \frac{m}{2}\sum_{i=1}^{N}\langle v_i^2\rangle\langle \boldsymbol{v}_j\rangle + \frac{m}{2}\left(\langle v_j^2\boldsymbol{v}_j\rangle - \langle v_j^2\rangle\langle \boldsymbol{v}_j\rangle\right) \tag{12}$$

In Eq. (12) the first term on the right-hand side is summed over N particles, and the second term is a one-particle correlation. Therefore, Eq. (12) can be written as

$$\langle E(v)\boldsymbol{v}\rangle \equiv \langle E(v)\boldsymbol{v}_j\rangle = \frac{m}{2}\sum_{i=1}^{N}\langle v_i^2\rangle\langle \boldsymbol{v}_j\rangle = \langle E(v)\rangle\langle \boldsymbol{v}_j\rangle \equiv \langle E(v)\rangle\langle \boldsymbol{v}\rangle$$

The equality $\langle Ev^2\rangle = \langle E\rangle\langle v^2\rangle$ can be proven in a similar way, which together with the first result will immediately give

$$\langle E[\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{r}, t)]\rangle = 0 \tag{13}$$

$$\langle E(v)[\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{r}, t)]^2\rangle = \langle E\rangle[\langle v^2\rangle - \langle v\rangle^2] = \langle E\rangle(\delta v)^2 \tag{14}$$

Substitution of the equalities in Eqs. (13) and (14) into Eq. (11) gives

$$-\partial_t\langle E(\boldsymbol{r}, t)\rangle = -\tfrac{1}{2}\beta^*(\langle E\rangle/T)\boldsymbol{u}\cdot\nabla T - (1-\alpha)\langle E\rangle(\nabla\cdot\boldsymbol{u}) \tag{15}$$

Because regions at higher T are lighter in particle density one expects, on the average, the $\boldsymbol{u}$ to be in the direction of grad T, which is opposite to the particle gradient. Then $\Delta\boldsymbol{a} = \Delta a\hat{n}$ and $\boldsymbol{u} = u\hat{n}$. Therefore, Eq. (8) together with Eq. (15) will give

$$\int_0^{\tau}\partial_t\langle \boldsymbol{J}(\boldsymbol{r}, \theta, t)\rangle\,\mathrm{d}\theta + \langle \boldsymbol{J}(\boldsymbol{r}, t, t)\rangle = -\lambda\nabla T - \gamma\hat{n}(\nabla\cdot\boldsymbol{u}) \tag{16}$$

Equation (16) can be written as

$$\tau\left(\frac{1}{\tau}\int_0^{\tau}\partial_t\langle \boldsymbol{J}(\boldsymbol{r}, \theta, t)\rangle\,\mathrm{d}\theta\right) + \langle \boldsymbol{J}(\boldsymbol{r}, t, t)\rangle = -\lambda\nabla T - \gamma\hat{n}(\nabla\cdot\boldsymbol{u}) \tag{17}$$

The expression

$$\frac{1}{\tau}\int_0^{\tau}\partial_t\langle \boldsymbol{J}(\boldsymbol{r}, \theta, t)\rangle\,\mathrm{d}\theta$$

is identified with the time average, and using the ergodic theory, it is equated to the ensemble average. Therefore, Eq. (17) is written as

$$\tau\frac{\partial\langle \boldsymbol{J}(\boldsymbol{r}, t)\rangle}{\partial t} + \langle \boldsymbol{J}(\boldsymbol{r}, t, t)\rangle = -\lambda\nabla T - \gamma\hat{n}(\nabla\cdot\boldsymbol{u}) \tag{18}$$

which is the Cattaneo's heat flux equation with an extra term that vanishes for fluids that can be classified as incompressible. Note that in λ, if one sets $\Delta a = N\pi d^2$ and sets $\langle E\rangle = Nmc_vT$, then $\lambda = \frac{1}{2}\beta^* n\ell mc_v u$, where $1/N\pi d^2 = n\ell$. The preceding equation for λ can be immediately identified with the thermal conductivity, with β^* as the Eucken number. For a monoatomic gas $\langle v^2\rangle = 3k_BT(\boldsymbol{r}, t)/m$ and using the average velocity of a monoatomic gas $u = (8k_BT(\boldsymbol{r}, t)/\pi m)^{1/2}$, the velocity fluctuations will be given by $(\delta v)^2 = (3\pi - 8)k_BT(\boldsymbol{r}, t)/\pi m$. A direct substitution of this velocity fluctuations into the equation for β^* will immediately give $\beta^* = 2.547$ as our Eucken number.

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Effect of Insulating Cuffs on an Internally Heated Wire

C. Y. Wang*
Michigan State University,
East Lansing, Michigan 48824-1027

Introduction

THE Joule heating from electric wires is an essential design factor for electric heating elements[1] and transmission lines in electric power systems.[2] For a long cylindrical wire heated internally uniformly, and its outer surface kept at constant temperature, the temperature distribution can be shown to be parabolic radially.[3] However, if the resistance increases with temperature then the heating would not be uniform. Jakob[4,5] found that there exists a maximum radius above which a mutual increase in temperature and resistance would lead to possible thermal failure. Boundary conditions other than isothermal, but still invariant axially, were considered by Morgan[2] and Fagan and Leipziger.[6] Finite cylinders with isothermal end conditions were investigated by Preckshot and Gorman.[7] Douglass[8] studied a wire where one segment has lowered heat loss because of shielding. Ignoring radial variations, he solved separately and matched axially the temperature distributions for the different segments.

The present Note studies the three-dimensional effects of insulating bands or cuffs placed over an internally nonuniformly heated cylinder or wire. These attachments may be insulating supports that hold the wire in place. Because they are also poor thermal conductors, local temperature increases may be considerable.

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*Professor, Departments of Mathematics and Mechanical Engineering.

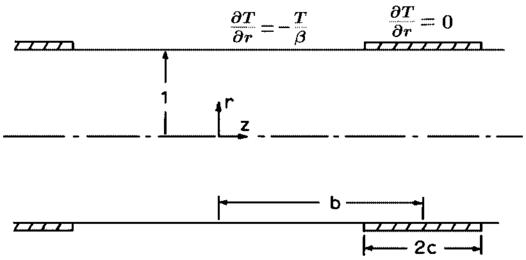

Fig. 1 Longitudinal cross section of a wire with cuffs.

The following assumptions will be used. The wire is homogeneous with constant conductivity. This is because conductivity varies by less than 2% with 100°C temperature rise.[2] The resistance, or internal heat generation, is the linear function of temperature.[4] The surface under the cuff is adiabatic, and on the exposed surface heat is transported uniformly by convection or weak radiation.

Formulation

The steady-state, axisymmetric heat conduction equation in cylindrical coordinates (r', z') is

$$\frac{\partial^2 T'}{\partial r'^2} + \frac{1}{r'}\frac{\partial T'}{\partial r'} + \frac{\partial^2 T'}{\partial z'^2} = -\frac{q_0}{k}[1 + \delta(T' - T_0)] \tag{1}$$

Here k is the thermal conductivity, $q_0 = i^2 R_0 / V$ is the Joule heating per volume V because of current i and resistance R_0 at an ambient temperature T_0. For pure metals the temperature coefficient δ is essentially constant.[2] We normalize all lengths by the wire radius R and drop primes. The temperature T' is related to the normalized temperature T by

$$T' = \left(q_0 R^2/k\right) T(r, z) + T_0 \tag{2}$$

Then Eq. (1) becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \alpha^2 T = -1 \tag{3}$$

where $\alpha^2 = \delta q_0 R^2/k$ is a nondimensional parameter representing the temperature coefficient. Let $2bR$ be the period between two cuffs and $2cR$ be the width of the cuffs. Figure 1 shows the origin of r, z is situated at the midpoint of the bare section. The temperature is symmetric, i.e., $\partial T/\partial z = 0$, at $z = 0$ and b and we need to consider only the region $0 \le z \le b$. The boundary conditions are that on the axis the temperature is bounded and on the bare surface:

$$\frac{\partial T}{\partial r}(1, z) = -\frac{1}{\beta} T(1, z) \qquad 0 \le z < b - c \tag{4}$$

where $\beta \equiv k/(Rh) = 1/(Nu)$. The parameter $\beta \to 0$ for an isothermal boundary and $\beta \to \infty$ for an adiabatic boundary. In forced convection cases β is of order unity. Under the cuff we assume little heat transfer:

$$\frac{\partial T}{\partial r}(1, z) = 0 \qquad b - c < z \le b \tag{5}$$

For given α, β, Eqs. (3–5) are to be solved.

If the cuff is absent ($c = 0$) the solution is then independent of z:

$$T(r) = \frac{1}{\alpha^2}\left[\frac{J_0(\alpha r)}{J_0(\alpha) - \beta\alpha J_1(\alpha)} - 1\right] \tag{6}$$

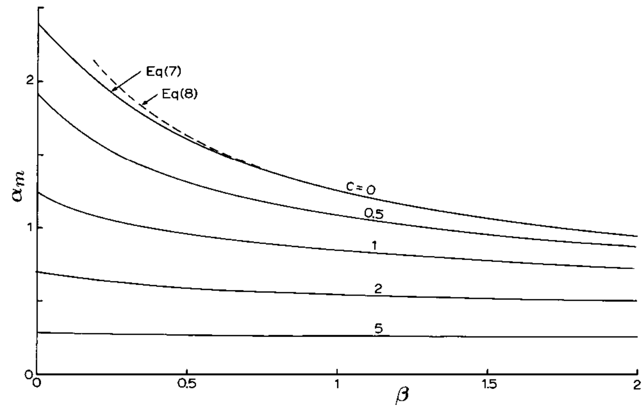

Fig. 2 Maximum α for a single cuff.

(Ref. 2). Note that the temperature becomes infinite when

$$J_0(\alpha) = \beta\alpha J_1(\alpha) \tag{7}$$

Here we give an asymptotic formula for the maximum α (or $\sqrt{(\delta q_0/k)}R$) from Eq. (7). Expanding the Bessel functions for small α, we obtain

$$\alpha_m \sim \sqrt{8/(1+4\beta)} + 0(\beta^{-2}), \qquad \beta \to \infty, \qquad c = 0 \tag{8}$$

Figure 2 shows Eq. (8) is valid for $\beta > 1$ as compared to the exact numerical values from Eq. (7). When $\beta = 0$ or isothermal conditions, Jakob's[4,5] value of $\alpha_m = 2.4048$ (first zero of J_0) is recovered.

Effect of Insulating Cuff

Attaching insulating cuffs (Fig. 1) impedes heat loss and increases the internal temperature unevenly. The general solution to Eq. (3), periodic in z, is

$$T(r, z) = -\frac{1}{\alpha_2} + A_0 J_0(\alpha r) + \sum_1^\infty A_n \cos(\lambda_n z) \begin{cases} e^{-\lambda_n} I_0(k_n r) & \alpha < \lambda_n \\ J_0(k_n r) & \alpha > \lambda_n \end{cases} \tag{9}$$

where A_n are coefficients to be determined, $\lambda_n \equiv n\pi/b$, and $k_n \equiv |\alpha^2 - \lambda_n^2|^{1/2}$. The boundary conditions Eqs. (4) and (5) give

$$-A_0\alpha J_1(\alpha) + \sum_1^\infty A_n \cos(\lambda_n z)\begin{cases} e^{-\lambda_n} k_n I_1(k_n) & \alpha < \lambda_n \\ -k_n J_1(k_n) & \alpha > \lambda_n \end{cases} = \begin{cases} -\dfrac{1}{\beta}\left[-\dfrac{1}{\alpha_2} + A_0 J_0(\alpha) + \displaystyle\sum_1^\infty A_n \cos(\lambda_n z)\begin{Bmatrix} e^{-\lambda_n} I_0(k_n) \\ J_0(k_n)\end{Bmatrix}\right] & \text{if } 0 \le z < b - c \\ 0 & \text{if } b - c < z \le b \end{cases} \tag{10}$$

We truncate the series to N terms. Then Eq. (10) is multiplied by $\cos(\lambda_m z)$, $m = 0$ to N, and integrated from $z = 0$ to b. The resulting $N + 1$ linear algebraic equations in the unknowns $A_0, \ldots, A_N$ can be inverted easily. The accuracy can be ascertained by increasing N. In general we find $N = 30$ is adequate for a three-digit accuracy in temperature.

Results and Discussion

The effect of a single insulating cuff on an internally heated wire is represented by the case $b \gg 1$, $c \approx \mathcal{O}(1)$. We find the interaction from neighboring cuffs is negligible when $b \sim 10$. Then using different values of β and c, the maximum possible α (α_m) is found from the method in the preceding section. Figure 2 shows α_m decreases with the Nusselt number $1/\beta$, but decreases more dramatically with increased cuff length $2c$. As $c \to \infty$, $\alpha_m \to 0$, or only very small diameter wires or very low currents can be supported by a wire with long cuffs. The maximum temperature T_m (at $r = 0$, $z = b \approx 20$) is shown in Fig. 3 for the single cuff, $\beta = 0.5$ and $\alpha < \alpha_m$. The